Proof of two conjectures of Z.-W. Sun on congruences for Franel numbers

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Abstract. For all nonnegative integers n, the Franel numbers are defined as

$$f_n=\sum_{k=0}^n\binom{n}{k}^3.$$

We confirm two conjectures of Z.-W. Sun on congruences for Franel numbers:

$$\sum_{k=0}^{n-1}(3k+2)(-1)^kf_k\equiv 0\pmod{2n^2},$$
$$\sum_{k=0}^{p-1}(3k+2)(-1)^kf_k\equiv 2p^2(2^p-1)^2\pmod{p^5},$$

where n is a positive integer and $p>3$ is a prime.

Keywords: Franel numbers, binomial coefficients, multinomial coefficients, congruences

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1 Introduction

The numbers f_n are defined to be the sums of cubes of binomial coefficients:

$$f_n=\sum_{k=0}^n\binom{n}{k}^3.$$

In 1894, Franel [3, 4] obtained the following recurrence relation for f_n:

$$(n+1)^2f_{n+1}=(7n^2+7n+2)f_n+8n^2f_{n-1},\ n=1,2,\dots. \tag{1.1}$$

Nowadays, the numbers f_n are usually called Franel numbers. The Franel numbers also appear in the first and second Strehl identities [12, 13] (see also Koepf [7, p. 55]):

$$\sum_{k=0}^n\binom{n}{k}^3=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{n},$$
$$\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^n\sum_{j=0}^k\binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3.$$

Applying the recurrence relation (1.1), Jarvis and Verrill [6] proved the following congruence for Franel numbers

$$f_n \equiv (-8)^n f_{p-1-n} \pmod{p},$$

where p is a prime and $0 \le n \le p-1$. Recently, Z.-W. Sun [15], among other things, proved several interesting congruences for Franel numbers, such as

$$\sum_{k=0}^{p-1} (-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1} (-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1} (-1)^k k^2 f_k \equiv \frac{10}{27}\left(\frac{p}{3}\right) \pmod{p^2},$$

where $p > 3$ is a prime and $\left(\frac{a}{3}\right)$ denotes the Legendre symbol. Sun [15] also proposed many amazing conjectures on congruences for f_n. The main purpose of this paper is to prove the following results, which were conjectured by Sun [15].

Theorem 1.1. *For any positive integer n, there holds*

$$\sum_{k=0}^{n-1} (3k+2)(-1)^k f_k \equiv 0 \pmod{2n^2}. \tag{1.2}$$

Theorem 1.2. *For any prime $p > 3$, there holds*

$$\sum_{k=0}^{p-1} (3k+2)(-1)^k f_k \equiv 2p^2(2^p-1)^2 \pmod{p^5}. \tag{1.3}$$

2 Proof of Theorem 1.1

We need the following identity due to MacMahon [8, p. 122] (see also Foata [2] or Riordan [10, p. 41]):

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n+k}{3k}\binom{3k}{2k}\binom{2k}{k} x^k (1+x)^{n-2k}. \tag{2.1}$$

When $x = 1$, the above identity (2.1) gives a new expression for Franel numbers:

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n+k}{3k}\binom{3k}{2k}\binom{2k}{k} 2^{n-2k}. \tag{2.2}$$

Differentiating both sides of (2.1) twice with respect to x and then letting $x = 1$, we get

$$\sum_{k=0}^{n}\binom{n}{k}^3 k(k-1) = \sum_{k=0}^{\lfloor n/2\rfloor} 2^{n-2k}\binom{n+k}{3k}\binom{3k}{2k}\binom{2k}{k}\frac{n(n-1)-2k}{4}. \tag{2.3}$$

Moreover, by induction, we can easily prove that

$$\sum_{\ell=2k}^{n-1}(-1)^\ell(3\ell+2)\binom{\ell+k}{3k}2^{\ell-2k} = (-1)^{n-1}(n-2k)\binom{n+k}{3k}2^{n-2k}. \tag{2.4}$$

In fact, when $n = 2k$, both sides of (2.4) are equal to 0. Now suppose that (2.4) is true for n. Then

$$\begin{aligned}
&\sum_{\ell=2k}^{n}(-1)^\ell(3\ell+2)\binom{\ell+k}{3k}2^{\ell-2k} \\
&= (-1)^n(3n+2)\binom{n+k}{3k}2^{n-2k} + \sum_{\ell=2k}^{n-1}(-1)^\ell(3\ell+2)\binom{\ell+k}{3k}2^{\ell-2k} \\
&= (-1)^n(3n+2)\binom{n+k}{3k}2^{n-2k} + (-1)^{n-1}(n-2k)\binom{n+k}{3k}2^{n-2k} \\
&= (-1)^n(n-2k+1)\binom{n+k+1}{3k}2^{n-2k+1},
\end{aligned}$$

which implies that (2.4) holds for $n+1$.

Applying (2.2) and then exchanging the summation order, we have

$$\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k = (-1)^{n-1}\sum_{k=0}^{n-1}2^{n-2k}(n-2k)\binom{n+k}{3k}\binom{3k}{2k}\binom{2k}{k} \tag{2.5}$$

in view of (2.4). Noticing that

$$n-2k = 4\frac{n(n-1)-2k}{4} - (n^2-2n),$$

by (2.2) and (2.3), we can write the right-hand side of (2.5) as

$$\begin{aligned}
&(-1)^{n-1}4\sum_{k=0}^{n}\binom{n}{k}^3 k(k-1) + (-1)^n(n^2-2n)\sum_{k=0}^{n}\binom{n}{k}^3 \\
&\quad = (-1)^{n-1}4n^2\sum_{k=0}^{n}\binom{n}{k}\binom{n-1}{k-1}^2 + (-1)^n n^2\sum_{k=0}^{n}\binom{n}{k}^3,
\end{aligned}$$

where we have used the following relations:

$$k\binom{n}{k} = n\binom{n-1}{k-1},$$

$$\sum_{k=0}^{n}k\binom{n}{k}^3 = \sum_{k=0}^{n}(n-k)\binom{n}{k}^3 = \frac{n}{2}\sum_{k=0}^{n}\binom{n}{k}^3.$$

Namely, we have proved that

$$\frac{1}{2n^2}\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k = (-1)^{n-1}2\sum_{k=0}^{n}\binom{n}{k}\binom{n-1}{k-1}^2 + (-1)^n\frac{f_n}{2}. \tag{2.6}$$

The proof then follows from the fact

$$f_n = \sum_{k=0}^{n}\binom{n}{k}^3 \equiv \sum_{k=0}^{n}\binom{n}{k} = 2^n \equiv 0 \pmod 2,\ n \geq 1.$$

By (2.2), for $n \geq 1$, we have

$$f_n \equiv \begin{cases} 2, & \text{if } n \text{ is a power of } 2, \\ 0, & \text{otherwise}, \end{cases} \pmod 4.$$

In fact, if $n = 2m+1 \geq 3$ is odd, then $\binom{2k}{k}2^{2m-2k+1} \equiv 0 \pmod 4$ for all $k \leq m$ and so $f_{2m+1} \equiv 0 \pmod 4$; if $n = 2m$ is even, then $f_{2m} \equiv \binom{3m}{2m}\binom{2m}{m} = 2\binom{3m}{2m}\binom{2m-1}{m} \pmod 4$ and the result follows from the congruences:

$$\binom{2m-1}{m} \equiv \begin{cases} 1, & \text{if } m \text{ is a power of } 2, \\ 0, & \text{otherwise}, \end{cases} \pmod 2$$

and $\binom{3m}{2m} \equiv 1 \pmod 2$ if m is a power of 2.

Thus, we may further refine Theorem 1.1 as follows:

Theorem 2.1. *For any positive integer n, there holds*

$$\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k \equiv \begin{cases} 2n^2, & \text{if } n \text{ is a power of } 2, \\ 0, & \text{otherwise}, \end{cases} \pmod{4n^2}. \tag{2.7}$$

3 Proof of Theorem 1.2

The following lemma is due to Sun [15, Lemma 2.1].

Lemma 3.1 (Sun)**.** *For any prime $p > 3$, there holds*

$$f_{p-1} \equiv 1 + 3(2^{p-1}-1) + 3(2^{p-1}-1)^2 \pmod{p^3}. \tag{3.1}$$

To prove Theorem 1.2, we also need the following variation of Lemma 3.1.

Lemma 3.2. *For any prime $p > 3$, there holds*

$$\sum_{k=1}^{p-1}\binom{p-1}{k}\binom{p-1}{k-1}^2 \equiv 2^{p-1} - 2^{2p-2} \pmod{p^3}. \tag{3.2}$$

Proof. It easily follows from cubing $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$ that

$$\sum_{k=0}^{p} \binom{p-1}{k}^3 = \sum_{k=0}^{p} \binom{p}{k}^3 - \sum_{k=0}^{p} \binom{p-1}{k-1}^3 - 3\sum_{k=0}^{p} \binom{p}{k}\binom{p-1}{k}\binom{p-1}{k-1}. \tag{3.3}$$

Since $\binom{p}{k} \equiv 0 \pmod p$ for $0 < k < p$, we have

$$\sum_{k=0}^{p} \binom{p}{k}^3 \equiv 2 \pmod{p^3}. \tag{3.4}$$

Substituting (3.1) and (3.4) into (3.3), we immediately get

$$\sum_{k=0}^{p} \binom{p}{k}\binom{p-1}{k}\binom{p-1}{k-1} \equiv 2^p - 2^{2p-1} \pmod{p^3}. \tag{3.5}$$

On the other hand, replacing k by $p-k$, we obtain

$$\sum_{k=1}^{p-1} \binom{p-1}{k}\binom{p-1}{k-1}^2 = \sum_{k=1}^{p-1} \binom{p-1}{k}^2\binom{p-1}{k-1} = \frac{1}{2}\sum_{k=0}^{p} \binom{p}{k}\binom{p-1}{k}\binom{p-1}{k-1}. \tag{3.6}$$

Combining (3.5) and (3.6), complete the proof. □

Proof of Theorem 1.2. By (2.6) and (3.4), we have

$$\begin{aligned}
\frac{1}{2p^2}\sum_{k=0}^{p-1}(3k+2)(-1)^k f_k &\equiv 2\sum_{k=0}^{n} \binom{p}{k}\binom{p-1}{k-1}^2 - 1, \\
&= 2\sum_{k=0}^{n} \binom{p-1}{k}\binom{p-1}{k-1}^2 + 2\sum_{k=0}^{n}\binom{p-1}{k-1}^3 - 1.
\end{aligned}$$

The proof then follows from (3.1) and (3.2). □

4 Concluding remarks and open problems

For any nonnegative integers n and r, let

$$f_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^r.$$

Then $f_n^{(3)} = f_n$ are the Franel numbers. Calkin [1, Proposition 3] proved the following congruence:

$$f_n^{(2r)} \equiv 0 \pmod p,$$

where p is a prime such that $\frac{n}{m} < p < \frac{n+1}{m} + \frac{n+1-m}{m(2mr-1)}$ for some positive integer m. Guo and Zeng [5, Theorem 4.4] proved that, for any positive integer n,

$$f_n^{(2r)} \equiv 0 \pmod{n+1}. \tag{4.1}$$

Sun [14, Conjecture 3.5] conjectured that

$$\sum_{k=0}^{n-1}(3k+2)f_k^{(4)} \equiv 0 \pmod{2n}. \tag{4.2}$$

It is easy to see that $f_n^{(0)} = n+1$, $f_n^{(1)} = 2^n$, $f_n^{(2)} = \binom{2n}{n}$ by the Chu-Vandermonde identity (see [7, p. 41]), and

$$\sum_{k=0}^{n-1}(3k+2)f_k^{(0)} = n^3 + n^2, \tag{4.3}$$

$$\sum_{k=0}^{n-1}(-1)^k(3k+2)f_k^{(1)} = (-1)^{n-1}2^n n, \tag{4.4}$$

$$\sum_{k=0}^{n-1}(3k+2)f_k^{(2)} = n\binom{2n}{n}. \tag{4.5}$$

Motivated by the identities (4.3)–(4.5), the congruence (1.2), and Sun's conjecture (4.2), we would like to propose the following conjecture on congruences for $f_n^{(r)}$.

Conjecture 4.1. *Let $n \geq 1$ and $r \geq 0$ be two integers. Then*

$$\sum_{k=0}^{n-1}(-1)^{rk}(3k+2)f_k^{(r)} \equiv 0 \pmod{2n}. \tag{4.6}$$

By (4.1), if the congruence (4.6) holds, then we have

$$\sum_{k=0}^{n-1}(3k+2)f_k^{(2r)} \equiv 0 \pmod{n(n+1)}.$$

For example, the first values of $\sum_{k=0}^{n-1}(3k+2)f_k^{(6)}$ are

$$2, 12, 540, 16600, 784500, 35315784, 1772807064, 90283679280, 4777960538340,$$

while $\sum_{k=0}^{n-1}(-1)^k(3k+2)f_k^{(5)}$ gives

$$2, -8, 264, -5104, 132460, -3373824, 91312256, -2513335808, 70719559668.$$

It seems that, for $n > 1$, the following congruence holds:

$$\sum_{k=0}^{n-1}(-1)^k(3k+2)f_k^{(2r+1)} \equiv 0 \pmod{4n}.$$

Recall that the multinomial coefficients are given by

$$\binom{n}{k_1,\ldots,k_m} = \frac{n!}{k_1!\cdots k_m!},$$

where $k_1,\ldots,k_m \geq 0$ and $k_1+\cdots+k_m = n$. Let

$$M_{m,n}^{(r)} = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\ldots,k_m}^r$$

be the sums of rth powers of multinomial coefficients. Then $M_{m,n}^{(0)} = \binom{m+n-1}{n}$, $M_{m,n}^{(1)} = m^r$, $M_{1,n}^{(r)} = 1$, $M_{2,n}^{(r)} = f_n^{(r)}$, and

$$M_{3,n}^{(2)} = \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}^2 \binom{k}{j}^2 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Note that Osburn and Sahu [9] studied supercongruences for the numbers $\sum_{k=0}^{n} \binom{n}{k}^r \binom{2k}{k}^s$. The sequence $\{M_{3,n}^{(2)}\}_{n\geq 0}$ is the A002893 sequence of Sloane [11]. It also appears in Zagier [16, #8 of Table 1]. Sun [15] proved the following identity involving $M_{3,k}^{(2)}$:

$$\sum_{k=0}^{n-1}(4k+3)M_{3,k}^{(2)} = 3n^2 \sum_{k=0}^{n-1} \frac{1}{k+1}\binom{2k}{k}\binom{n-1}{k}^2. \tag{4.7}$$

It seems that Conjecture 4.1 can be further generalized as follows.

Conjecture 4.2. *Let $m,n \geq 1$ and $r \geq 0$ be integers. Then*

$$\sum_{k=0}^{n-1}(-1)^{rk}((m+1)k+m)M_{m,k}^{(r)} \equiv 0 \pmod{mn}. \tag{4.8}$$

It is not hard to prove the following identities by induction.

$$\sum_{k=0}^{n-1}((m+1)k+m)M_{m,k}^{(0)} = mn\binom{m+n-1}{m}, \tag{4.9}$$

$$\sum_{k=0}^{n-1}(-1)^k((m+1)k+m)M_{m,k}^{(1)} = (-1)^{n-1}m^n n, \tag{4.10}$$

$$\sum_{k=0}^{n-1}(2k+1)M_{1,k}^{(2r)} = n^2, \tag{4.11}$$

$$\sum_{k=0}^{n-1}(-1)^k(2k+1)M_{1,k}^{(2r+1)} = (-1)^{n-1}n. \tag{4.12}$$

Combining the congruence (1.2), the identities (4.5), (4.7) and (4.9)–(4.12), we have the following result to support Conjecture 4.2.

Proposition 4.3. *The congruence* (4.8) *holds if* (m, r) *belongs to*

$$\{(m, 0), (m, 1), (1, r), (2, 2), (2, 3), (3, 2) : m, r \geq 1\}.$$

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References

[1] N.J. Calkin, Factors of sums of powers of binomial coefficients, Acta Arith. 86 (1998), 17–26.

[2] D. Foata, Etude algébrique de certains problèmes d'analyse combinatoire et du calcul des probabilités, Publ. Inst. Statist. Univ. Paris 14 (1965), 81–241.

[3] J. Franel, On a question of Laisant, L'intermédiaire des mathématiciens 1 (1894), 45–47.

[4] J. Franel, On a question of J. Franel, L'intermédiaire des mathématiciens 2 (1895), 33–35.

[5] V.J.W. Guo and J. Zeng, New congruences for sums involving Apéry numbers or central Delannoy numbers, Int. J. Number Theory, to appear.

[6] F. Jarvis and H. A. Verrill, Supercongruences for the Catalan-Larcombe-French numbers, Ramanujan J. 22 (2010), 171–186.

[7] W. Koepf, Hypergeometric Summation, an Algorithmic Approach to Summation and Special Function Identities, Friedr. Vieweg & Sohn, Braunschweig, 1998.

[8] P. A. MacMachon, Combinatorial Analysis, Vol. 1, Cambridge University Press, London, 1915.

[9] R. Osburn and B. Sahu, Supercongruences for Apéry-like numbers, Adv. Appl. Math. 47 (2011), 631–638.

[10] J. Riordan, Combinatorial Identities, J. Wiley, New York, 1979.

[11] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, available at `http://www.research.att.com/~njas/sequences/`

[12] V. Strehl, Binomial sums and identities, Maple Technical Newsletter 10 (1993), 37–49.

[13] V. Strehl, Binomial identities — combinatorial and algorithmic aspects, Discrete Math. 136 (1994), 309–346.

[14] Z.-W. Sun, Conjectures and results on x^2 mod p^2 with $4p = x^2 + dy^2$, In: Number Theory and the Related Topics, Eds., Y. Ouyang, C. Xing, F. Xu and P. Zhang, Higher Education Press & International Press, Beijing and Boston, 2012, to appear.

[15] Z.-W. Sun, Congruences for Franel numbers, preprint, arXiv:1112.1034v6.

[16] D. Zagier, Integral solutions of Apéry-like recurrence equations, In: J. Harnad, P. Winternitz, Eds., Group and Symmetries: From Neolithic Scots to John McKay, CRM Proceedings & Lecture Notes, 47, American Mathematical Society, Providence, RI, 2009, pp. 349–366.